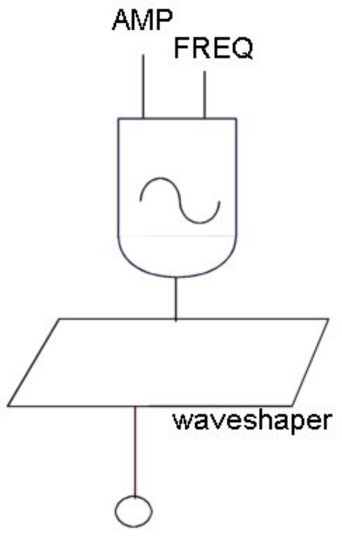
Waveshaping Synthesis

Waveshaping is a *waveform distortion*-based technique. It provides an economic way to synthesise complex time-evolving spectra, requiring only a few unit generators.

Waveshaping is based on a system with non-linear response, which will distort the amplitude of an input waveform, shaping it in some way. The main element that accomplishes this is a *non-linear processor* or *waveshaper*, which will distort the input signal according to its *transfer function*.

Another incarnation of waveshaping is found in *overdrivers* and *fuzz-boxes*, where an input is distorted, resulting in the addition high-frequency components and a bright timbre

The Basic Instrument



A waveshaping instrument can be built simply with a sinewave oscillator driving the waveshaper.

The function of the waveshaper is to map the input values in a non-linear form, so to distort the input wave.

In a linear processor, the output would be equal to the input, because its transfer function is a straight line.

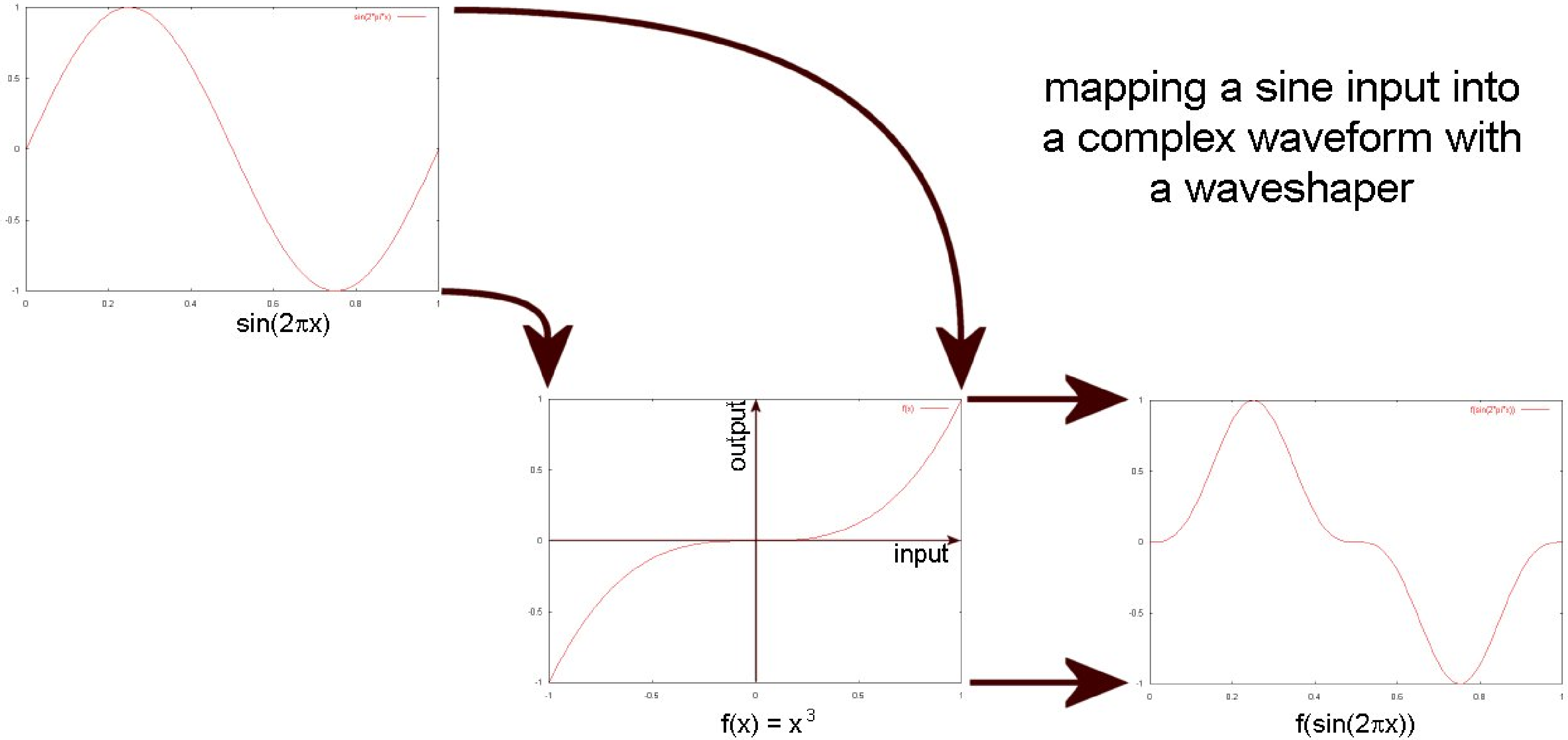
A waveshaper will produce an output that will be different to its input, depending on the shape of the *transfer function*,

Transfer Functions and Mapping

The mapping process takes each sample from the input signal and find its corresponding output value by looking up a transfer function (usually stored in a table).

If say, the input is $sin(x)$, the transfer function $f(x)$, then the output will be $f(sin(x))$. It is easy to see that if $f(x)$ is linear, as in $f(x) = 0.5x$, the output will have the same shape as the input (although, here, with half the amplitude).

If the transfer function is non-linear, as in $f(x) = x^3$, we will have some sort of amplitude distortion in the output. Usually the amount of distortion will vary with the amplitude of the input signal, but that depends on the shape of the transfer function.



This example shows how an input is mapped into the output using a non-linear transfer function. Notice how the input and output waveshapes differ from each other.

Transfer Function Shapes

Transfer functions can be defined by their shape. In general, the following characteristics will be determined by their shape:

1. A *straight line* produces *no* amplitude distortion.
2. The more *extreme* the change in *slope*, the *richer* the spectrum. Smoother functions will produce less components.
3. An *even function* (symmetric about the y-axis) produces only *even harmonics*.
4. An *odd function* (anti-symmetrical about the y-axis) produces only *odd harmonics*.
5. Transfer function shapes with *discontinuities* or *sharp points* will generate an *infinite* number of harmonics, some of which will be *aliased*.

Transfer functions and Polynomials

In order to avoid aliasing and have a more precise idea of the output spectra, we usually define a transfer function using a polynomial.

A polynomial is defined by an expression that is a sum of one or more terms containing a variable raised to different powers. The order of the polynomial is determined by the highest power term in the expression.

A transfer function will be non-linear if it is of order 2 or higher. Examples:

$f(x) = x^2 + x$ $\quad g(x) = 2x^5 + 0.1x^3$ $\quad h(x) = 7x^{10} + 2x^8 + 4x^3$

Band-limited spectra

The advantage of using polynomials for transfer functions is that they guarantee band-limited spectra. This can in turn be used to avoid aliasing.

When driven with a sinusoidal input, a waveshaper with a polynomial transfer function of order N will not produce components above the Nth harmonic.

Another advantage is that, by using a polynomial function, we can very easily determine the output spectra in terms of the different amplitudes of the various components.

Output Spectra and Transfer Functions

A simple table (taken from Pascal's triangle) is used to calculate the harmonic content of an output signal of a waveshaper with a polynomial transfer function driven by a cosine wave.

On this table:

• each line is associated with a particular *polynomial term* and a *divisor*.

• the column values are *harmonic weights*; they are divided by the associated *divisor*, which provides the *harmonic contribution* of that *term* to the output.

• the contribution of *all terms* for *each harmonic* is added up; this determines the spectrum of the sound when the *driving cosine* has amplitude *1*.

The Table

	DIV	h0	h1	h2	h3	h4	h5	h6	h7
x^0	0.5	1							
x^1	1		1						
x^2	2	2		1					
x^3	4		3		1				
x^4	8	6		4		1			
x^5	16		10		5		1		
x^6	32	20		15		6		1	
x^7	64		35		21		7		1

Exs: $\boldsymbol{f(x) = x^3}$ => 1 term:

h1 = 3/4, **h3** = 1/4 [**h1** is harm 1 (fund), **h3** is harm 3]

$\boldsymbol{f(x) = x^5+x^3+x^2+x}$ => 4 terms:

(1) **h1** = 1;

(2) **h0** = 2/2, **h2** = 1/2; [**h0** is the *0Hz* component, or *DC offset*)

(3) **h1** = 3/4, **h3** = 1/4;

(4) **h1** =10/16, **h3** = 15/16, **h5** = 1/16

output spectrum: **h0**=1, **h1**=2.375, **h2**=0.5, **h3**=1.1875, **h5**=0.0625

Characteristics of polynomial transfer functions

It can be seen from the table that:

- a even-numbered term will contribute only to even harmonics.
- an odd-numbered term will contribute only to odd ones.

Lower input amplitudes (< 1) will generate less spectral spread and higher amplitudes (> 1), more.

It is important to note that the contribution of each term will be *multiplied* by the respective *power* of the input amplitude. In this case the contribution of term 2 (x^2) will be multiplied by the square of the amplitude, term 3 (x^3) by its cube, etc..

By controlling the amplitude of the driving sinusoid, we will be able to control the richness of the output spectrum. This amplitude is referred to as the *distortion index*.

Spectral Matching

One of the ways to tackle the selection of a suitable transfer function is by *spectral matching*.

This is a technique that creates transfer functions to generate a certain steady-state spectrum at a certain *distortion index*.

A special kind of polynomials, known as *Chebyshev Polynomials* of the first kind, can be used for that purpose.

These polynomials have a special property that, when a driving ***cosine*** wave has amplitude one, a waveshaper transfer function using a Chebyshev polynomial of order *k* will generate an output that only contains the *k*th harmonic.

Chebyshev Polynomials

Using these types of polynomials to generate a transfer function, we can generate any desired steady-state spectrum. This is done by combining two or more polynomials in one expression.

The Chebyshev polynomials $T_k(x)$, of order $k=0$ and $k=1$, are 1 and x, respectively; the general formulation for any higher-order Chebyshev polynomial is:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

So,

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Spectral Matching example

In order to realise a steady state, using 4 harmonics (1,3,5,7) with amplitudes of 1, 0.5, 0.6, 0.2, we could proceed as follows:

(1) Combine 4 Chebyshev polynomials of orders 1,3,5,7:

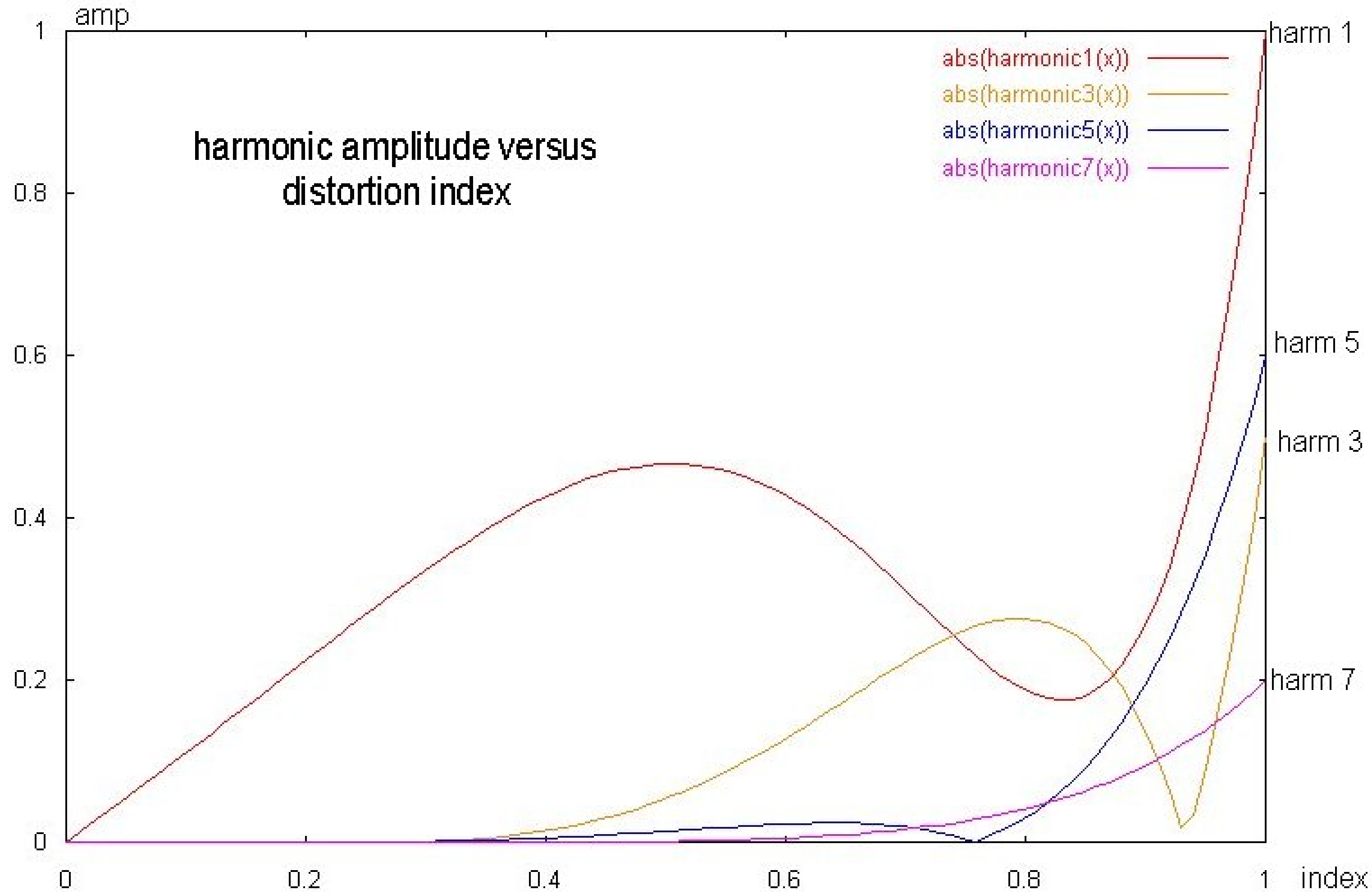
$$F(x) = x + (4x^3 - 3x) + (16x^5 - 20x^3 + 5x) + (64x^7 - 112x^5 + 56x^3 - 7x)$$

(2) Multiply each polynomial by then respective harmonic amplitude:

$$F(x) = x + 0.5(4x^3 - 3x) + 0.6(16x^5 - 20x^3 + 5x)$$
$$+ 0.2(64x^7 - 112x^5 + 56x^3 - 7x)$$

(3) Work it out into a single polynomial expression:

$$F(x) = 12.8x^7 - 12.8x^5 + 1.2x^3 + 1.1x$$



Spectral evolution, for indices between 0 and 1, of waveshaper output with transfer function $F(x) = 12.8x^7 - 12.8x^5 + 1.2x^3 + 1.1x$

The *moduli* of each harmonic amplitude is plotted against the index.

Transfer function selection considerations

Spectral matching is perhaps the easiest way to select a suitable transfer function. In addition, other methods can be used:

- *graphical*: directly drawing the transfer function. However, this will not guarantee that the output will be bandlimited, so aliasing can occur.

- *polynomial coefficients* can also be selected directly. Care should be taken to choose the signs of terms, as a fully positive/negative transfer function can produce very bright timbres. Alternating the signs for the even/odd-order terms can be a suitable solution for smoother results.

Spectral evolution

One of the problems with waveshaping is that there is an intrinsic difficulty in producing smooth spectral evolutions.

It is often the case that a polynomial that produces a good, balanced, steady-state timbre will exhibit ripples in the evolution of harmonics with distortion index.Generally, the higher the polynomial order, the harder it is to produce smooth spectral changes.

With spectral matching, one trick is to select the sign of the even and odd harmonic amps to be alternated independently. For instance, in the case of harmonics 1,2,3,4 and 5 with amps 1, 0.5, 0.3, 0.25 and 0.2, we would choose the signs 1, -0.5, -0.3, 0.25, 0.2. The overall pattern is +, +, -, -, +, +, ..., starting from h0.

Implementing a waveshaping instrument

A waveshaper is implemented by using a table lookup procedure. The table will contain the transfer function, centred at 0 and generally defined within the interval -1 and 1.

The table lookup will be normalised (0, 1) and offset by 0.5, so that index 0 is actually read at the table mid-point.

In this case we will drive the waveshaper with an oscillator containing a cosine wave whose max amplitude will be 0.5, when the distortion index is 1 (a max range of 1, between -0.5 and 0.5).

```
acos  oscili   indx*0.5, ifreq, 1, 0.25 ; cosine signal
awsh  tablei   acos, 2, 0.5, 1          ; waveshaper
```

GEN functions for waveshaping

Any GEN drawing a bipolar function centered on the table mid-point can be used in waveshaping. However two GENs are specialised for it:

GEN 03 => draws a polynomial over any interval (in our case -1 to 1), from specified coefficients.

```
f1 0 4097 3 -1 1 0 1.1 0 1.2 0 -12.8 0 12.8
```

draws $F(x) = 12.8x^7 - 12.8x^5 + 1.2x^3 + 1.1x$ over -1 : 1

GEN 13 => draws a Chebyshev polynomial matching a given spectrum

```
f1 0 4097 13 -1 1 0 1 0 0.5 0 0.6 0 0.2
```

draws the same polynomial as the previous example.

Amplitude Scaling Considerations

Another characteristic of waveshaping instruments is that the *distortion index* also determines the amplitude of the output signal. The larger the index, the richer the spectrum and the louder the sound.

This might be a desirable feature: many real instruments have a coupling between timbral and amplitude envelopes. However, it is better to be able to control the timbre and the amplitude evolutions separately.

This is achieved with an amplitude envelope placed at the output of the waveshaper. This will also require that we *scale* the output of the waveshaper so that it does not vary wildly with the distortion index.

Transfer function normalisation

A solution is to scale the output of a waveshaper so that it produces the same level no matter what the distortion index is.

This is done by using a normalisation function, which will be a function of the distortion index. The normalisation function can be created directly from the transfer function.

A scaling factor is then obtained from this function, by table lookup indexed *directly* by the distortion value. The output of the instrument is the product of the scaling factor and the waveshaper output:

$$output = F(a\cos(x)) \times S(a)$$

where *a* is the *distortion index* and *S()* is the *normalising function*.

Normalising function generation

In csound, it is very easy to create a normalising function for a certain transfer function.

GEN 4 reads the contents of a function table and creates a table with a normalising function.

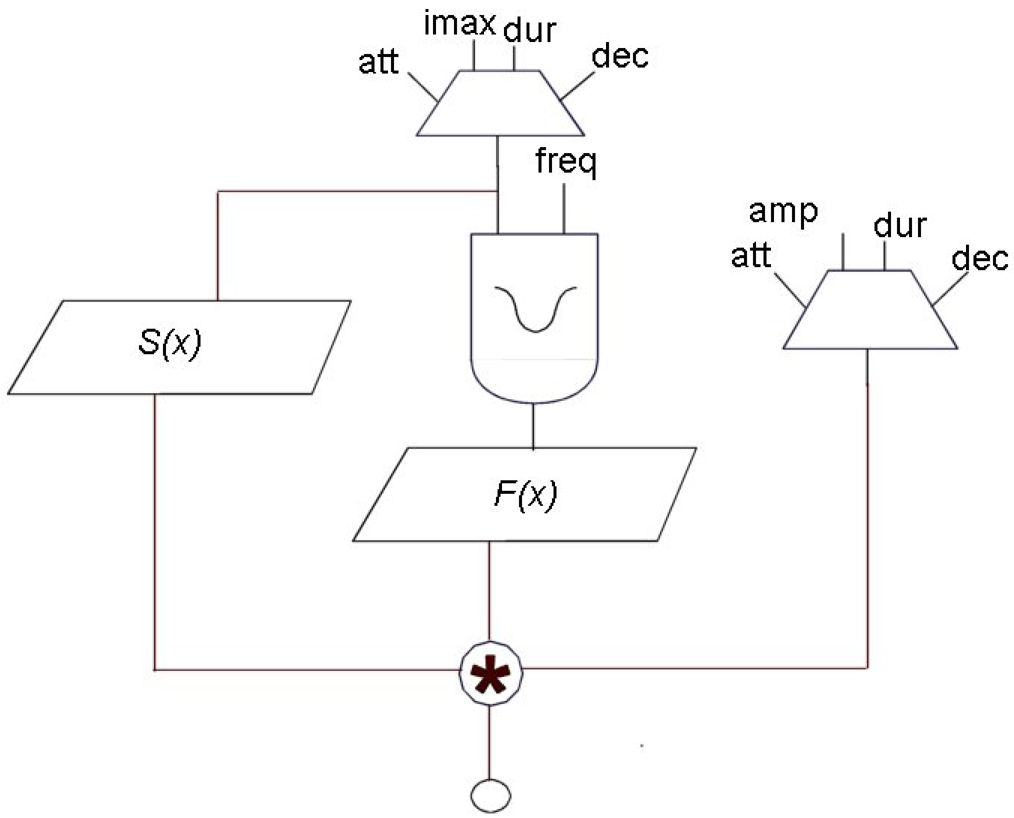
Here, the transfer function is bipolar (from -1 to +1), so GEN 4 starts from the table mid-point and proceeds outwards examining pairs of values to create the normalising function. Because of this, the table size is going to be always 1/2 the transfer function size.

```
f2  0  2049  4  1  1
```

Generates a normalising function for ftable 1, which contains a transfer function of size **4097** (4096 with ext. guard point).

A basic waveshaping instrument

A basic waveshaping instrument with timbral & amplitude envelopes would look like this:



Csound waveshaping instrument code

```
iamp = p4
ifr  = p5
imax = p6
icos  = 1    /* cosine wave ftable */
itrns = 2    /* transfer ftable    */
iscal = 3    /* normalising ftable */

kamp   linen  iamp, 0.05, p3,.4  /* amp envelope */
kndx   linen  imax, 0.1, p3, .4  /* timbre envelope */
acos   oscili kndx*0.5, ifr, icos /* driving oscillator */
awsh   tablei acos, itrns, 1, 0.5 /* waveshaper  */
kscl   tablei kndx, iscal, 1      /* scaler      */

       out    kscl*awsh*kamp
```

Waveshapers with Ring Modulation

Waveshapers produce harmonic spectra, when driven by a sinusoid and with a suitable transfer function.

With a simple variation, using ring modulation, we will be able to realise a larger variety of sounds, including inharmonic spectra.

The principle is to multiply the output of the waveshaper by a sinusoid oscillator with a certain frequency f_2 and amplitude A_2. This will produce a signal with a spectrum composed of the sum and difference of all frequencies of the shaped wave with f_2:

$$f_2 \pm jf_{shaped}, \text{ where } j = 0,1,2\ldots N$$

$$\text{with amps} \Rightarrow 0.5 \times A_2 \times h_j$$

N is the order of the transfer polynomial

h_j is the jth waveshaper harmonic amplitude

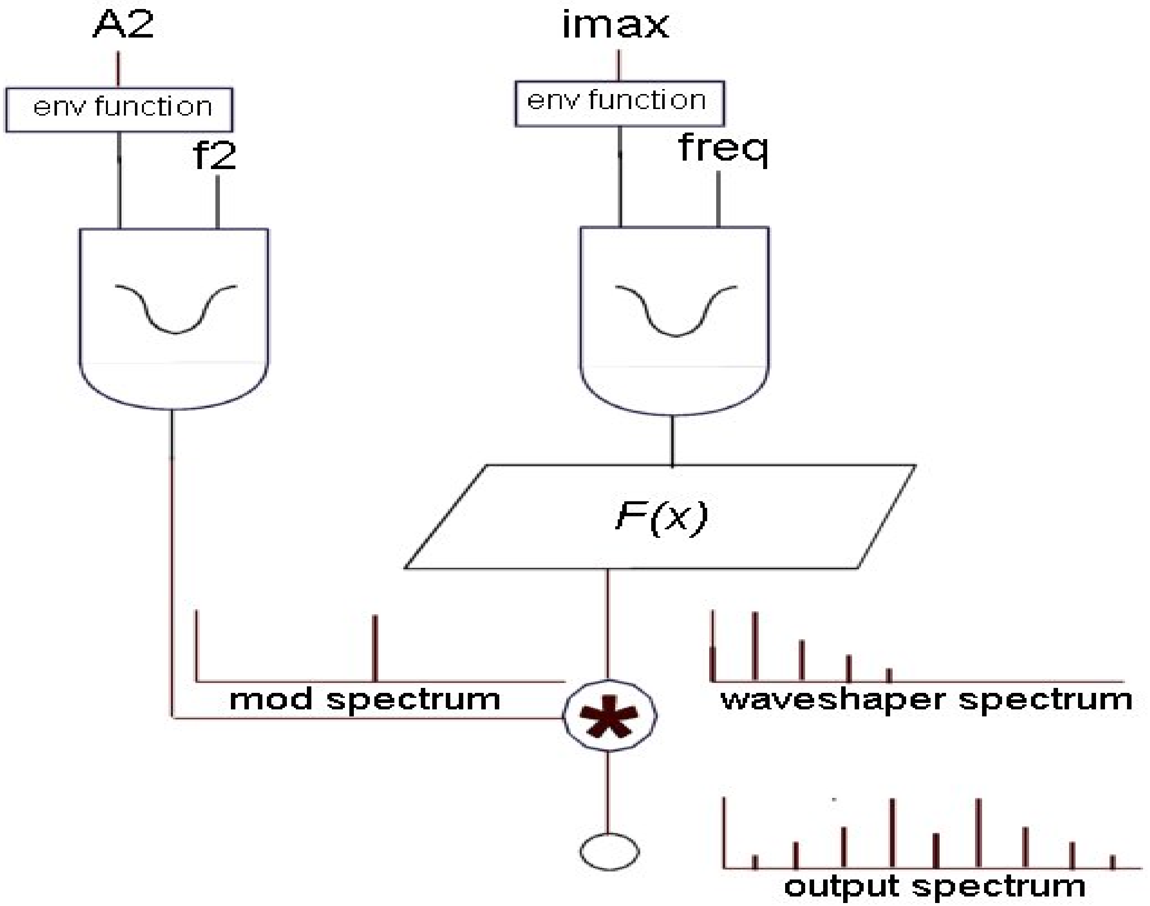
Similarities to FM spectra

This arrangement produces an output with similarities to FM synthesis. A number of sidebands around the modulator frequency are produced.

The ratio between *f2* and *fshaped* will determine the fundamental or whether the spectrum is harmonic or inharmonic:

$$\text{if } \frac{f_{shaped}}{f_2} = \frac{N_1}{N_2} \text{ then } f_0 = \frac{f_{shaped}}{N_1} = \frac{f_2}{N_2}$$

where N_1 and N_2 are integers with no common factors. When N_1 is even, only odd harmonics are generated. If f_2 or f_{shaped} is irrational, then an inharmonic spectrum results.



A waveshaper+ring modulation instrument and the resulting sound spectrum. The modulator amplitude *A2* controls the overall sound intensity. Envelopes are used for timbre & amplitude

Multiple Waveshapers

The previous technique can be extended with the use of multiple waveshapers. This would involve the combination of two (or more) waveshapers with one modulating oscillator.

The resulting spectrum will be composed of the sum and difference of all components of both waveshapers and modulating sinusoid:

$$f_{mod} \pm jf_{shaped1} \pm kf_{shaped2},$$

where j and k are the orders of the two polynomials

Here many components will be generated even with low order transfer functions. The advantage of this is that lower-order polynomials generally provide smoother spectral evolutions.

Beauchamp's Cornet

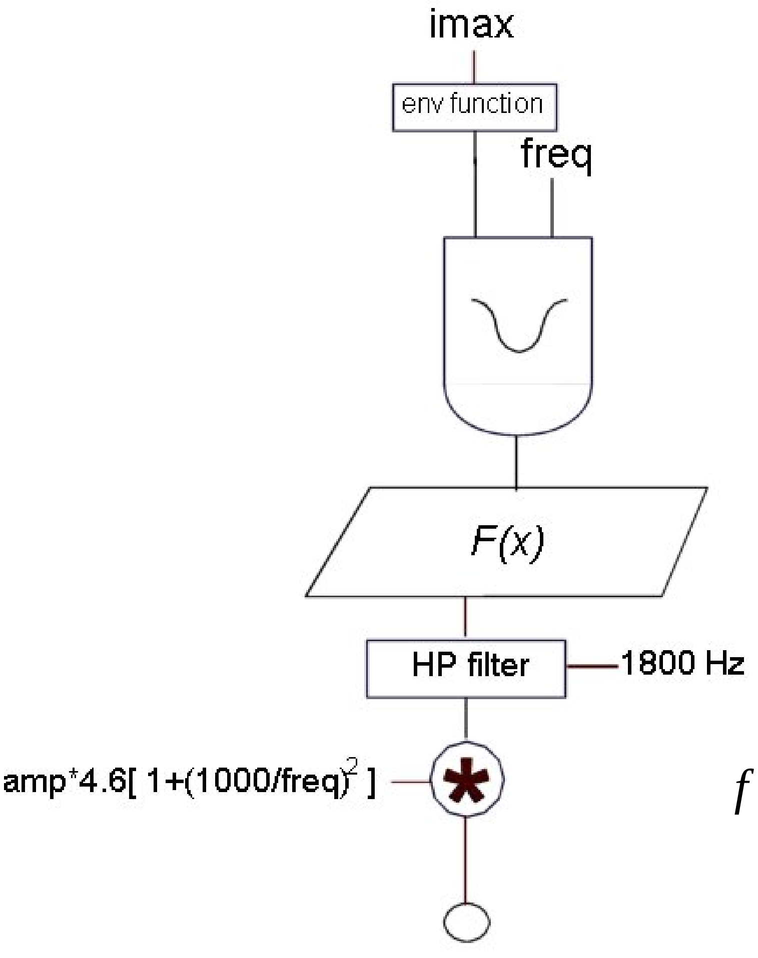
Another interesting example of waveshaping applications was demonstrated by James Beauchamp.

His design models a cornet by:

(a) using a waveshaper to approximate the output of an instrument mouthpiece, using a spectral-matched transfer function.

(b) using a filter to model the resonator body of the instrument, which exhibits a high-pass characteristic.

Cornet emulation instrument design



Beauchamp's design uses different envelope functions for different amplitudes.

The HP filter used is a 2nd order Butterworth filter.

The mouthpiece transfer function is (over -1.4:1.4):

$$
\begin{aligned}
f(x) = {} & .03667x^{12} + .02791x^{11} - .09983x^{10} \\
& - .07557x^{9} + .11342x^{8} + .08414x^{7} \\
& - .06547x^{6} - .02972x^{5} + .6308x^{4} \\
& + .02060x^{3} - .00508x^{2} + .03052x
\end{aligned}
$$

Summary

- Waveshaping instruments are based on distortions of an input signal.
- A *waveshaper* is a *non-linear processor* that maps its input into an output according to a certain transfer function.
- *Transfer functions* can be created using polynomials. The order of the polynomial determines the highest component, when driven by a sinusoid.
- *Chebyshev polynomials* can be used to generate transfer functions matching a particular steady-state spectrum.
- *Amplitude scaling* should be considered if an independent spectral envelope is needed.
- Waveshapers are implemented using *table lookup* ugens, or table readers.
- With the addition of *ring modulation*, a larger variety of sounds can be created by a waveshaping instrument.